Migration of the Separation Point on a Deforming Cylinder

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An iterative scheme of solving the Navier-Stokes equations for unsteady two-dimensional flow about a deforming and translating cylinder is given. In the *k*th iteration, the convected vorticity appears as the source term in an equation of transient diffusion of vorticity. A novel integral transform is used to reduce this transient vorticity equation to a *k*-dimensional heat equation. The bounded solution of this equation is obtained with a general method of superposition for problems involving a moving boundary. An equation describing the migration of the separation point on a deforming cylinder in unsteady cross flows is derived from the analytically obtained velocity field. The radial expansion of the cylinder surface is shown to hasten the separation time and to increase the separation angle. The results imply that in the steady flow past a body that is moving forward and sinking at constant rates, the locus of points at which the azimuthal component of skin friction changes sign originates on the leeward ray at a point downstream of the front tip.

I. Introduction

THE unsteady flow separation about a radially deforming cylinder starting an arbitrary translation from rest is analyzed. This problem is relevant to several fundamental issues of considerable importance. These issues include the control of flow separation by moving an aerodynamic surface,[1] the possible drag reduction by surface deformation,[2] and the issue on the appropriate definition of an unsteady flow separation point.[3] However, this study was originally motivated by the need to understand the nature of the large-scale vortical flow that may develop on the leeward side of a body of revolution that is moving forward and sinking at constant rates. An observer watching the flow in a vertical plane that is fixed in space perpendicular to the axis of the body, and through which the body passes, sees the evolution of a sectionally two-dimensional vortical flow structure about a circle whose radius changes with time. Thus, there is a certain analogy, called the "impulsive-flow analogy," between the steady three-dimensional flow about a slender body of revolution and the unsteady two-dimensional flow about a circular cylinder whose radius changes with time in a cross flow. Topological features of the analogy have been demonstrated by Tobak and Peak[4] and Chapman and Tobak.[5]

Creeping flow around a deforming sphere was studied by Lin and Gautesen[2] on the basis of the unsteady Stokes equation. The linear unsteady Stokes equation is known also to be a valid first-order approximate equation even at finite Reynolds numbers for small-amplitude high-frequency fluctuating flows,[6-8] as well as for initial transient flows.[9] The nonlinear convective acceleration terms were later treated as source terms in perturbation solutions to the Navier-Stokes equations by Tseng and Lin[10,11] in their study of transient heat transfer from a heated wire and for constructing a theory of a heat-sensing velocimeter. While these perturbation solutions are useful for describing the initial transience of flows in which the convective acceleration is weaker than the local acceleration, they are obviously inadequate for the description of separate flows in which the convective and local accelerations may become equally important at a later stage in the flow development.

An iterative scheme for solving the Navier-Stokes equations for unsteady two-dimensional flow about a deforming and translating cylinder is given in the next section. To the best knowledge of the writers, this method is novel and cannot be found in any treatises on the Navier-Stokes equations.[12] First, we take the curl of the Navier-Stokes equations to obtain the vorticity equation. This equation represents the balance between the linear transient diffusion of vorticity and the nonlinear convection of vorticity. In unsteady flows starting from rest, the diffusion of vorticity dominates the convective transport of vorticity during the initial transience.[6-9] Thus, in the first approximation, the nonlinear convective terms are neglected. The bounded solution of the resulting linear equation of vorticity diffusion is then obtained. The obtained solution is then substituted into the previously neglected nonlinear terms, which now serve as the distributed sources for the transient vorticity diffusion equation. The solution of this equation constitutes the first iterated solution. The same procedure can be repeated again and again to yield successively higher iterations. The transient vorticity diffusion equation with a source distribution in the *k*th iteration is shown to be reducible to a *k*-dimensional heat equation. The bounded solution of this equation with the appropriate boundary conditions is given in Sec. III. A general method of superposition for problems involving a moving boundary is given in the Appendix. An equation that describes the migration of the separation point on a deforming cylinder is obtained in Sec. IV. The numerical results are then presented and discussed. The present iterative solution is expected to be valid during the initial transience before a large amount of vorticity is convected to infinity. Then the well-known difficulty associated with the Whitehead paradox[13] surfaces. The numerical results show that the flow separation establishes itself long before the vorticity has time to populate itself at infinity.

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II. Formulation

Consider the flow of an incompressible Newtonian fluid about a circular cylinder. The fluid is initially quiescent. The translational velocity of the cylinder is $-\boldsymbol{i}U(t)$, where t is time and $\boldsymbol{i}$ is the unit vector in the negative x_1 direction, as shown in Fig. 1. The instantaneous radius of the deforming cylinder is given by $a_1(t)$.

The governing equations with respect to a reference frame attached to the cylinder are

$$\nabla \cdot V = 0$$

$$\partial_t V + V \cdot \nabla V = -(1/\rho)\nabla P + \nu \nabla^2 V + i\dot{U}(t) - jg \tag{1}$$

where V is the velocity, P the pressure, ρ density, ν the kinematic viscosity, and jg the gravitational force per unit mass of fluid. Let the maximum of $U(t)$ be U_m and the maximum of $a_1(t)$ be a_0. Nondimensionalizing the distance, time, velocity and pressure, respectively, by a_0, a_0^2/ν, U_m, and ρU_m^2, Eq. (1) can be written as

$$\partial_\tau v + R_e(v \cdot \nabla)v = -R_e \nabla p + \nabla^2 v \tag{2a}$$

$$\nabla \cdot v = 0 \tag{2b}$$

where

$$v = V/U_m \qquad \tau = t/[a_0^2/\nu]$$

$$p = (P + \rho a_0 g y - \rho a_0 \dot{U}(t)x)/\rho U_m^2$$

$$R_e = U_m a_0/\nu$$

Equation (2b) is the necessary and sufficient condition for the existence of the stream function ψ, such that the radial and tangential velocity components of two-dimensional velocity can be written, respectively, as

$$v_r = -(1/r)\partial_\theta \psi \qquad v_\theta = \partial_r \psi$$

In terms of the stream functions, Eqs. (2a) and (2b) can be written as

$$(\partial_\tau - \nabla^2)\nabla^2 \psi = -\frac{R_e}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r,\theta)} = -\frac{R_e}{r} J \tag{3}$$

where

$$\nabla^2 = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$$

$$J = \partial_r \psi \partial_\theta \nabla^2 \psi - \partial_\theta \psi \partial_r \nabla^2 \psi$$

The initial condition is

$$\psi(r,\theta,\tau) = 0, \qquad \tau \le 0$$

The boundary conditions are

$$-\left(\frac{1}{r}\partial_\theta \psi\right)_{r=a(\tau)} = \frac{1}{R_e}\frac{da}{d\tau} = \frac{1}{R_e}\dot{a}, \quad (\partial_r \psi)_{r=a(\tau)} = 0 \tag{4}$$

$$-\left(\frac{1}{r}\partial_\theta \psi\right)_{r\to\infty} = u(\tau)\cos\theta, \qquad (\partial_r \psi)_{r\to\infty} = -u(\tau)\sin\theta \tag{5}$$

where $u(\tau) = U(t)/U_m$.

The iterative solution of Eqs. (3-5) will be obtained for the case of flows that are symmetric with respect to the x axis. The generalization of the method for the case of asymmetric flows will be mentioned later. The mth iterated solution will be written as

$$\psi^{(m)} = \sum_{k=1}^{m} \psi_k^{(m)} \sin k\theta - a\dot{a}\theta/R_e \tag{6}$$

Substituting Eq. (6) into the Jacobian in Eq. (3), we have

$$J^{(m-1)} = \partial(\psi^{(m-1)}, \nabla^2\psi^{(m-1)})/\partial(r,\theta)$$

$$= \sum_{q=1}^{m-1}\sum_{\ell=1}^{m-1}\{A_{q\ell}^{(m-1)}\sin q\theta\cos\ell\theta - B_{q\ell}^{(m-1)}\cos q\theta \sin\ell\theta\} + \sum_{\ell=1}^{m-1} C_\ell^{(m-1)}\sin\ell\theta \tag{7}$$

where

$$A_{q\ell}^{(m-1)} = (\partial_r \psi_q^{(m-1)})(\ell D_\ell^2 \psi_\ell^{(m-1)})$$

$$C_\ell^{(m-1)} = a\dot{a}\partial_r D_\ell^2 \psi_\ell^{(m-1)}/R_e$$

$$B_{q\ell}^{(m-1)} = (q\psi_q^{(m-1)})(\partial_r D_\ell^2 \psi_\ell^{(m-1)}), \qquad m \ge 2$$

$$D_\ell^2 = \partial_{rr} + r^{-1}\partial_r - \ell^2 r^{-2}, \qquad J^{(0)} = 0$$

By use of the relation $\sin q\theta\cos\ell\theta = [\sin(q+\ell)\theta + \sin(q-\ell)\theta]/2$, we can rewrite Eq. (7) as

$$J^{(m-1)} = \sum_{\ell=1}^{m-1} C_\ell^{(m-1)}\sin\ell\theta + \frac{1}{2}\sum_{q=1}^{m-1}\sum_{\ell=1}^{m-1}\{(A_{q\ell}^{(m-1)} - B_{q\ell}^{(m-1)})\sin(q+\ell)\theta + (A_{q\ell}^{(m-1)} + B_{q\ell}^{(m-1)})\sin(q-\ell)\theta\} \tag{8}$$

Equation (8) can be rearranged into a form of truncated Fourier series

$$J^{(m-1)} = \sum_{k=1}^{m-1} C_k^{(m-1)}\sin k\theta + \frac{1}{2}\sum_{k=2}^{2m-2}\sum_{\ell=1}^{k-1}(A_{(k-\ell)\ell}^{(m-1)} - B_{(k-\ell)\ell}^{(m-1)})\sin k\theta \cdot h_{k(\ell+m)} \cdot h_{\ell(m)} + \frac{1}{2}\sum_{k=1}^{m-2}\sum_{\ell=1}^{m-1-k}(A_{(k+\ell)\ell}^{(m-1)} + B_{(k+\ell)\ell}^{(m-1)})\sin k\theta + \frac{1}{2}\sum_{k=1}^{m-2}\sum_{q=1}^{m-1-k}(A_{q(q+k)}^{(m-1)} + B_{q(q+k)}^{(m-1)})\sin(-k\theta) \tag{9}$$

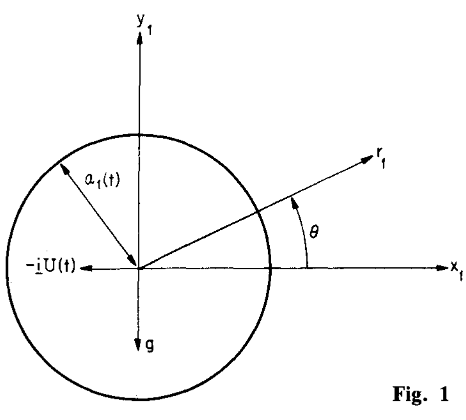

Fig. 1 Definition sketch.

where

$$h_{k(m)} = 1 \text{ if } k<m, \qquad \delta_{k1} = 1 \text{ if } k=1$$
$$= 0 \text{ if } k\geq m, \qquad = 0 \text{ if } k\neq 1$$

Equation (9) can be further reduced to

$$J^{(m-1)}=\sum_{k=1}^{m}\Big\{C_k^{(m-1)}h_{k(m)}+\frac{1}{2}\sum_{\ell=1}^{k-1}\Big[(1-\delta_{k1})(A_{(k-\ell)\ell}^{(m-1)}-B_{(k-\ell)\ell}^{(m-1)})$$
$$+\sum_{\ell=1}^{m-1-k}(A_{(\ell+k)\ell}^{(m-1)}+B_{(\ell+k)\ell}^{(m-1)})h_{k(m-1)}$$
$$-\sum_{\ell=1}^{m-1-k}(A_{\ell(\ell+k)}^{(m-1)}+B_{\ell(\ell+k)}^{(m-1)})h_{k(m-1)}\Big]\Big\}\sin k\theta$$
$$+\frac{1}{2}\sum_{k=m+1}^{2m-2}\sum_{\ell=1+k-m}^{m-1}(A_{(k-\ell)\ell}^{(m-1)}-B_{(k-\ell)\ell}^{(m-1)})h_{(k-\ell)(m)}\sin k\theta \tag{10}$$

We assume that the coefficients of $\sin k\theta$ with $k>m$ are negligibly small in the $(m-1)$th iteration, and write $J^{(m-1)}$ as

$$J^{(m-1)}=\sum_{k=1}^{m}J_k^{(m-1)}\sin k\theta \tag{11}$$

where $J_k^{(m-1)}$ is given by the sum of terms in the curly bracket in Eq. (10). Thus, the last double summation term in Eq. (10), which represents higher harmonics, will be assumed to be negligible. This assumption is based on the physical observation that during the initial transience the higher harmonics are generated and dominated by the lower harmonics and thus need not be considered in the lower-order iterative approximation.

Substituting Eqs. (6) and (11), respectively, into the left and right sides of Eq. (3), and equating the coefficients of the same harmonics on both sides, we have

$$(\partial_\tau-D_k^2)D_k^2\psi_k^{(m)}=-r^{-1}R_eJ_k^{(m-1)} \qquad (k\geq 1) \tag{12}$$

By use of the following transform:

$$\psi_k^{(m)}=u(\tau)(a^2r^{-1}-r)\delta_{k1}+r^{-k}\int_a^r\chi_k^{(m)}(s,\tau)s^{2k-1}\mathrm{d}s \tag{13}$$

Eq. (12) can be reduced to the $2k$-dimensional heat equation

$$(\partial_\tau-\nabla_k^2)\chi_k^{(m)}=G_k^{(m-1)} \tag{14}$$

where

$$G_k^{(m-1)}=-\int_{a(\tau)}^{r}s^{-k}J_k^{(m-1)}(s,\tau)\mathrm{d}s$$

$$\nabla_k^2=r^{1-2k}\partial_r(r^{2k-1}\partial_r)$$

Note that Eq. (6) with Eq. (13) satisfies the first condition of Eq. (4). In order to satisfy the second condition of Eq. (4), we must have

$$\chi_k^{(m)}[a(\tau),\tau]=2u(\tau)a^{1-k}(\tau)\delta_{k1} \tag{15}$$

Substituting the bounded solution of Eq. (14) with Eq. (15) into the boundary condition Eq. (5), we have

$$-\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)_{r\to\infty}=u(\tau)\cos\theta+f(\tau)\cos k\theta$$

$$\left(\frac{\partial\psi}{\partial r}\right)_{r\to\infty}=-u(\tau)\sin\theta+g(\tau)\sin k\theta$$

where $f(\tau)$ and $g(\tau)$ are the "penalty functions" given by

$$f(\tau)=\lim_{r\to\infty}\left[r^{-k-1}\int_a^r\chi_k^{(m)}(s,\tau)s^{2k-1}\mathrm{d}s\right]$$

$$g(\tau)=\lim_{r\to\infty}\left[-kr^{-k-1}\int_a^r\chi_k^{(m)}(s,\tau)s^{2k-1}\mathrm{d}s+\chi_k^{(m)}(r,\tau)r^{k-1}\right]$$

Thus, the bounded solution of Eq. (14) with Eq. (15) will not satisfy the boundary condition (5) unless $f(\tau)\to 0$ and $g(\tau)\to 0$. For the special case of harmonic oscillation with $m=k=1$, it was shown by Lin[6] that $f(\tau)\to 0$ and $g(\tau)\to 0$. In general, it is expected that there exists a small time τ_c below which $f(\tau)$ and $g(\tau)$ remain much smaller than $u(\tau)$. This expectation is based on the physical observation that during the initial transience, the vorticity does not have enough time to build up sufficiently to alter $u(\tau)$ at infinity. In any event, this expectation can be verified rigorously a posteriori.

III. Solution

The solution of Eq. (14) with its boundary condition Eq. (15) can be achieved as follows. First, the solution of Eq. (14), which satisfies the quasi-steady boundary condition corresponding to Eq. (15), will be obtained. This solution can be obtained from

$$(\partial_\tau-\nabla_k^2)\chi_k^{(m)}(r,\ell,\tau)=G_k^{(m-1)}(r,\ell,\tau) \tag{16}$$

$$\chi_k^{(m)}[a(\ell),\ell,\tau]=2u(\ell)a^{1-k}(\ell)\delta_{k1}H(\tau) \tag{17}$$

where ℓ is the time parameter associated with $a(\ell)$ and where $H(\tau)$ is the Heaviside unit step function, i.e.,

$$H(\tau)=1, \qquad \tau>0$$
$$=0, \qquad \tau\leq 0$$

It is shown in the Appendix that the solution of Eq. (14) with the unsteady boundary condition is then given by

$$\chi_k^{(m)}(r,\tau)=\int_0^\tau\partial_\tau\chi_k^{(m)}[r,\ell,\tau-\ell]\,\mathrm{d}\ell \tag{18}$$

Equation (16) can be solved by use of the Laplace transform. The Laplace transform of Eq. (16) is

$$[v^2-d_r^2-(2k-1)r^{-1}d_r]\bar{\chi}_k^{(m)}=\bar{G}_k^{(m-1)} \tag{19}$$

where v^2 is the Laplace transform variable. By use of the new variables

$$\bar{\chi}_k^{(m)}=S_k^{(m)}/r^{k-1} \quad \text{and} \quad \zeta=vr \tag{20}$$

Eq. (19) can be reduced to the nonhomogeneous modified Bessel equation of the $(k-1)$th order,

$$\{d_\zeta^2+\zeta^{-1}d_\zeta-[1+(k-1)^2\zeta^{-2}]\}S_k^{(m)}=-\bar{G}_k^{(m-1)}r^{k-1}v^{-2} \tag{21}$$

The solution of this equation is given by

$$S_k^{(m)}=[b_k^{(m)}(v)+B_k^{(m)}(vr)]I_{k-1}(vr)$$
$$+[c_k^{(m)}(v)+C_k^{(m)}(vr)]K_{k-1}(vr) \tag{22}$$

where

$$B_k^{(m)}=-\int_\infty^r s^kK_{k-1}(vs)\bar{G}_k^{(m-1)}(s,v)\mathrm{d}s$$

$$C_k^{(m)}=\int_0^r s^kI_{k-1}(vs)\bar{G}_k^{(m-1)}(s,v)\mathrm{d}s$$

For the bounded solution, we chose $b_k^{(m)}(v)=0$. Taking the inverse Laplace transform of Eq. (20) and applying the convolution theorem, we have, from Eqs. (20) and (22),

$$\chi_k^{(m)} = r^{1-k}\left\{\int_0^\tau q_k^{(m)}(\ell,\lambda)H_k(r,\tau-\lambda)\mathrm{d}\lambda + \int_0^\tau\int_0^\infty V_k(r,s,\tau-\lambda)G_k^{(m-1)}(s,\lambda)s^k\mathrm{d}s\mathrm{d}\lambda\right\} \tag{23}$$

where $q_k^{(m)}$ is the inverse Laplace transform of $v^2K_{k-1} \times [a(\ell)v]c_k^{(m)}(v)$, i.e.,

$$q_k^{(m)} = \mathcal{L}^{-1}v^2K_{k-1}[a(\ell)v]c_k^{(m)}(v)$$

$$H_k(r,\tau) = \mathcal{L}^{-1}\frac{K_{k-1}(vr)}{K_{k-1}[a(\ell)v]v^2} = \bar{H}_k(r,\tau)H(\tau)$$

$$\bar{H}_k(r,\tau) = 1+\frac{2}{\pi}\int_0^\infty \exp[-u^2\tau] \times\frac{J_{k-1}(ur)Y_{k-1}[a(\ell)u]-Y_{k-1}(ur)J_{k-1}[a(\ell)u]}{J_{k-1}^2[a(\ell)u]+Y_{k-1}^2[a(\ell)u]}\frac{\mathrm{d}u}{u}$$

and V_k is the inverse Laplace transform of $I_{k-1}(vr) \times K_{k-1}(vs)$ and $I_{k-1}(vs)K_{k-1}(vr)$, i.e.,

$$V_k = (1/2\tau)I_{k-1}(rs/2\tau)\exp[-(r^2+s^2)/4\tau]H(\tau)$$

In order to satisfy the boundary condition (17), $q_k^{(m)}$ in Eq. (23) must satisfy the following integral equation

$$\int_0^\tau q_k^{(m)}[a(\ell),\lambda]H_k[a(\ell),\tau]\mathrm{d}\lambda = 2u(\ell)\delta_{k1} - \int_0^\tau\int_{a(\ell)}^\infty V_k[a(\ell),s,\tau-\lambda]G_k^{(m-1)}(s,\ell,\lambda)s^k\mathrm{d}s\mathrm{d}\lambda \tag{24}$$

Note that the lower integration limit in the last integral in Eq. (24) is changed from (0) to $a(\ell)$ since $G_k^{(m-1)}$ vanishes for $s\le a$. Recall that

$$G_k^{(m-1)}(r,\ell,\tau) = -\int_{a(\ell)}^r R_e s^{-k}J_k^{(m-1)}(s,\tau)\mathrm{d}s$$

The solution of Eq. (24) for $q_k^{(m)}$ can be easily obtained by differentiating both sides of Eq. (24) with τ,

$$\int_{-\infty}^\infty \partial_\tau q_k^{(m)}[a(\ell),\lambda]\bar{H}_k[a(\ell),\tau-\lambda][H(\lambda)-H(\lambda-\tau)]\mathrm{d}\lambda$$

$$=2u(\ell)\delta(\tau)\delta_{k1}-\partial_\tau\int_{-\infty}^\infty[H(\lambda)-H(\lambda-\tau)] \times\int_{a(\ell)}^\infty V_k[a(\ell),s,\tau-\lambda]G_k^{(m-1)}(s,\ell,\lambda)s^k\mathrm{d}s\mathrm{d}\lambda$$

where $\delta(\tau)$ is the Dirac delta function. Performing the differentiation and using the relations $\bar{H}_k[a(\ell),0]=1$, $V[a(\ell),s,0]=0$, we reduce this equation to

$$q_k^{(m)}[a(\ell),\tau] = 2u(\ell)\delta_{k1}\delta(\tau) - \int_0^\tau\int_{a(\ell)}^\infty \partial_\tau V_k[a(\ell),s,\tau-\bar{\lambda}]G_k^{(m)}(s,\ell,\bar{\lambda})s^k\mathrm{d}s\mathrm{d}\bar{\lambda} \tag{25}$$

It follows from Eq. (23) that

$$r^{k-1}\chi_k^{(m)}(r,\ell,\tau) = 2u(\ell)\delta_{k1}H_k(r,\tau) - \int_0^\tau\int_0^\lambda\int_{a(\ell)}^\infty \partial_\lambda V_k[a(\ell),s,\lambda-\bar{\lambda}] \times G_k^{(m-1)}(s,\ell,\bar{\lambda})H_k(r,\tau-\lambda)s^k\mathrm{d}s\mathrm{d}\bar{\lambda}\mathrm{d}\lambda + \int_0^\tau\int_{a(\ell)}^\infty V_k(r,s,\tau-\lambda)G_k^{(m-1)}(s,\ell,\lambda)s^k\mathrm{d}s\mathrm{d}\lambda \tag{26}$$

Substituting this into Eq. (18), we have, after some simple manipulation,

$$r^{k-1}\chi_k^{(m)}(r,\tau) = \int_0^\tau 2u(\ell)\partial_\tau H_k(r,\ell,\tau-\ell)\mathrm{d}\ell\delta_{k1} - \int_0^\tau\int_0^{\tau-\ell}\int_0^\lambda\int_{a(\ell)}^\infty \partial_\lambda V_k[a(\ell),s,\lambda-\bar{\lambda}] \times G_k^{(m-1)}(s,\ell,\bar{\lambda})\partial_\tau H_k(r,\ell,\tau-\ell-\lambda)\mathrm{d}s\mathrm{d}\bar{\lambda}\mathrm{d}\lambda\mathrm{d}\ell + \int_0^\tau\int_0^{\tau-\ell}\int_{a(\ell)}^\infty \partial_\tau V_k(r,s,\tau-\ell-\lambda)G_k^{(m-1)}(s,\ell,\lambda)s^k\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\ell \tag{27}$$

With $\chi_k^{(m)}(r,\tau)$ given by Eq. (27), the mth iterative solution given by Eqs. (6) and (13) is now complete. The convergence of the iterative solution (6) may be proved by showing that there exists a time τ_c below which $\psi_k^{(m)}\le M$, M being a bounded constant, for given R_e, $U(\tau)$, $a(\tau)$, and any m and $k\le m$. It is expected that τ_c will decrease as R_e increases. However, the exact proof is not yet available.

The extension of the present method to nonsymmetric flow is immediate. In place of Eq. (6), the solution now can be written as

$$\psi^{(m)} = -a\dot{a}\theta + \sum_{k=1}^m(\epsilon R)^{k-1}\psi_k^{(m)}\sin k\theta + \sum_{k=1}^m(\epsilon R)^{k-1}\phi_k^{(m)}\cos k\theta \tag{28}$$

The Jacobian in Eqs. (7) and (11) will have to be modified to include $\cos k\theta$ terms. A set of integral transforms similar to that given by Eq. (13) will then reduce the partial differential equations for $\psi_k^{(m)}$ and $\phi_k^{(m)}$ into a set of $2k$-dimensional heat equations. The solution of $\phi_k^{(m)}$ and $\psi_k^{(m)}$ by the Laplace transform method will follow the line described for the solution of $\psi_k^{(m)}$ given above.

IV. Separation Point

An appropriate definition of separation points in unsteady two-dimensional flows is of great current interest. For the discussions on the controversy and the relevant references, we refer the readers to the work of Ho.[3] Here, we are concerned with the separation points on a radially deforming cylindrical surface. With respect to this surface, the velocity components (v_r', v_θ') are given by

$$v_r' = v_r - \dot{a} \qquad v_\theta' = v_\theta$$

If we define a separation point at such a surface to the point at which the flow in the azimuthal direction has zero shear stress at the wall, or equivalent by zero surface vorticity, then we must have

$$\partial_r v_\theta' = 0 \quad \text{at} \quad r=a(\tau)$$

Since $v_\theta' = v_\theta$, this condition can be written as

$$(\partial_{rr}\psi)_{r=a} = 0$$

It follows from Eq. (6) that at the separation points we must have

$$\sum_{k=1}^{m} (\partial_{rr}\psi_k^{(m)})_{r=a} \sin k\theta(\tau) = 0 \tag{29}$$

where, with the aid of Eq. (13),

$$(\partial_{rr}\psi_k^{(m)})_{r=a} = \frac{2u(\tau)}{a(\tau)}\delta_{k1} - a^{k-2}\chi_k^{(m)}(a,\tau) + a^{k-1}(\partial_r\chi_k^{(m)})_{r=a} \tag{30}$$

It is easily verified that at $r=a(\tau)$ the last two integrals in Eq. (27) vanish, and Eq. (27) gives

$$a^{k-1}\chi_k^{(m)}[a(\tau),\tau] = 2u(\tau)\delta_{1k}$$

as is required by the boundary condition Eq. (15). Thus, the first two terms in Eq. (30) cancel each other. The last term in Eq. (30) can be obtained from Eq. (27) by a simple differentiation. Combination of Eqs. (29) and (30) then gives

$$\sum_{k=1}^{m} a^{k-1}(\partial_r\chi_k^{(m)})_{r=a} \sin k\theta = 0 \tag{31}$$

Several comments concerning the general feature of two-dimensional flow separation can now be made. In the limiting case of creeping flows, $R_e \to 0$ and Eq. (31) is reduced to

$$(\partial_r\chi_1^{(m)})_{r=a} \sin\theta = 0$$

Thus, the only points on the cylinder where shear stress vanishes are at the forward and rear "stagnation" points, respectively, at $\theta=\pi$ and $\theta=0$. For a moderately large R_e such that the second term in Eq. (31) is comparable to the first one, only the second iteration may be adequate to approximate the separated flow. Equation (31) then gives

$$\sin\theta(\partial_r\chi_1^{(2)} + 2a\partial_r\chi_2^{(2)}\cos\theta)_{r=a} = 0 \tag{32}$$

Thus, in addition to the stagnation points, there will be two symmetrically positioned primary separation points given by

$$\cos\theta = -\left(\frac{\partial_r\chi_1^{(2)}}{2a\cdot\partial_r\chi_2^{(2)}}\right)_{r=a} \tag{33}$$

Equation (33) has a solution only after a finite separation time is reached when the quotient in Eq. (33) becomes less than one in absolute value. At higher Reynolds numbers, higher harmonic terms in θ must be retained. This may allow secondary and further separation points to be obtained as the roots of the transcendental equation of order higher than two. It should be pointed out that if vortex shedding takes place, the flow is no longer symmetric, and separated flow must be based on Eq. (28). Asymmetric flow separation is not included in this study.

V. Results

To obtain numerical results, we used short-time expansions of Bessel functions appearing in the integrands of the analytical results given in the previous sections. For the evaluation of the multiple integrals, we used the M-point Gauss quadrature formula.[14] For each integration interval, the value of M was increased until the values of the given integral corresponding to the two successive values of M differ from each other by less than 0.001%. Time integrations have been carried out only up to a time when $f(\tau)/U(\tau)$ or $g(\tau)/U(\tau)$ reaches 0.05. All numerical computation was carried out with double precision on an IBM 4341.

To compare our iterative solution with some known experimental and theoretical results, we obtained numerical results for the case of a rigid cylinder first. The time-dependent angle of separation was determined from Eq. (33) for the case of an impulsively started uniform motion of a rigid cylinder. The results are given in Fig. 2, together with other known results. The finite separation times for the various values of R_e are all slightly larger than those given by Collins and Dennis[15] and Thoman and Szewczyk.[16] The difference could probably be reduced if we advance our approximation to the third iteration. However, it should be pointed out that the initial condition used in this study differs from others. The initial flow obtained from a boundary-layer approximation was used by Collins and Dennis to kick off their numerical computation. Thomas and Szewczyk and others who used pure numerical solutions assumed either that the flow was initially irrotational everywhere or at some finite distance from the cylinder. The initial condition used in this study is that the fluid is completely quiescent everywhere. The separation angle for a given R_e increases rapidly after the onset of separation and approaches an almost steady value within a small fraction of viscous diffusion

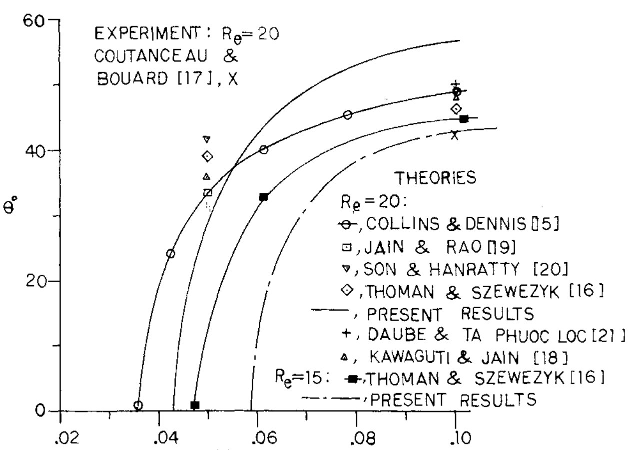

Fig. 2 Migration of separation point for the case of impulsively started uniform velocity.

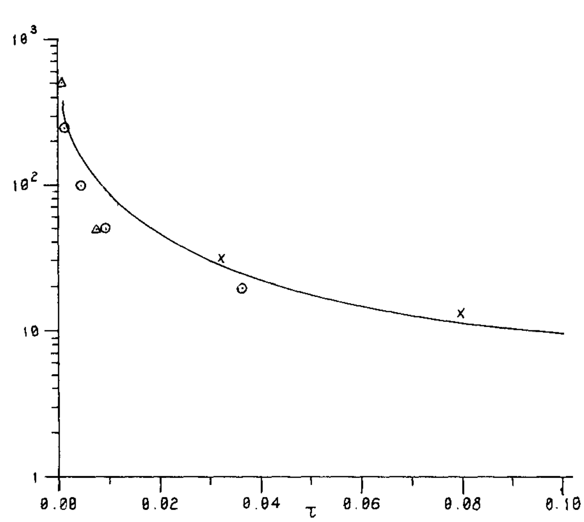

Fig. 3 Reynolds number (R_e) vs onset separation time (τ_s) x, Countanceau and Bouard[17]; ⊙ Collins and Dennis[15]; △, Wang[26]; —, present results; - - - -, Cowley.[23]

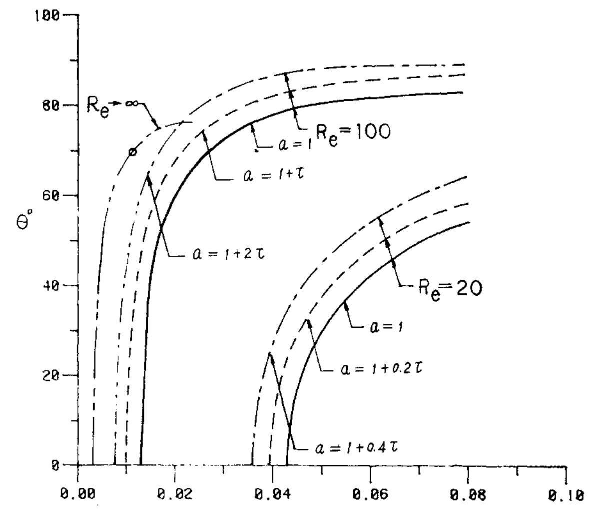

Fig. 4 Effect of radial expansion on separation angle for the case of impulsively started uniform velocity.

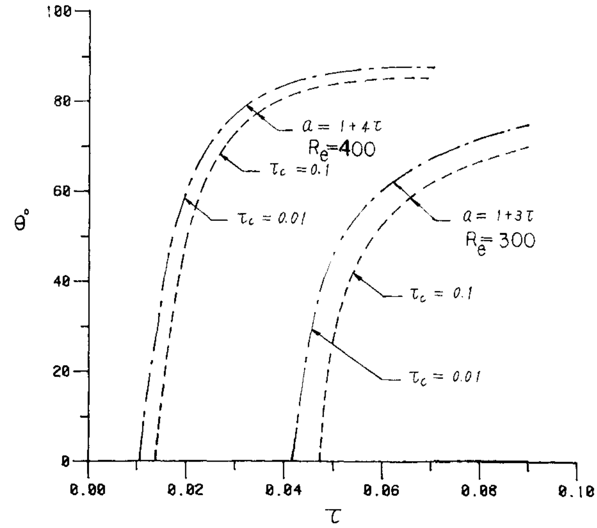

Fig. 6 Effects of acceleration on a deforming cylinder on separation angle.

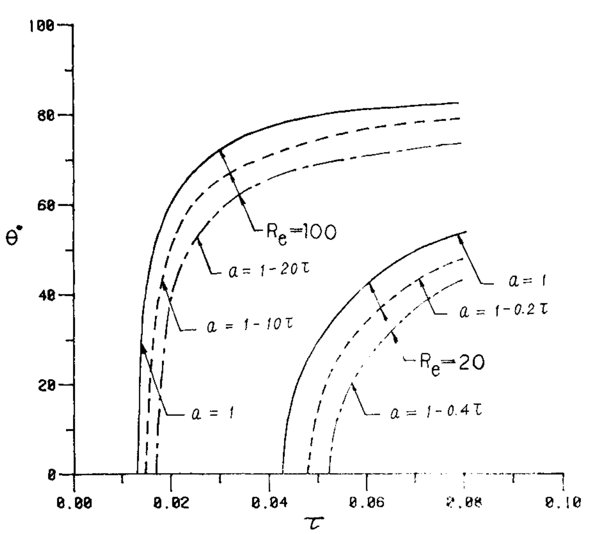

Fig. 5 Effect of radial contraction on separation angle.

time. Note that the dimensionless time t_1 of the quoted theoretical studies are related to the authors' by $\tau = t_1/R_e$. The authors are not aware of any measurements of separation angles for the initial time smaller than $\tau = 0.1$. We extrapolate the measurements of Coutanceau and Bouard[17] for $R_e = 20$ in a tank whose diameter is 8.3 times that of the test cylinder. Our predicted angle is considerably larger than their measured value. It is not clear if this is due to the wall effect. The wall of the test tank was shown by them to have the effect of reducing the separation angles. There is also considerable uncertainty involved in the extrapolation. It should be pointed out that the nondimensional time t^* of Coutanceau and Bouard is related to ours by $\tau = 2t^*/R_e$. Figure 3 shows that the separation time increases rapidly with decreasing R_e. The steep slope of the curve suggests that there is a critical R_e below which separation does not occur.

Figure 4 demonstrates the effect of the radial deformation of the cylinder, which impulsively starts a constant velocity translation, on the separation angle. It is seen that, for a given R_e, the radial expansion has an effect of reducing the onset time of separation and increasing the separation angle at the same instant. The larger the rate of expansion, the larger the effect. The earlier separation and large separation angle are all associated with large deceleration of fluid induced by radial expansion. The results obtained with the boundary-layer theory by Cowley[23] is also included in Fig. 4 for comparison. Their results agree with earlier results, including those of Dommelen and Shen.[24] The circle on the curve indicates the approximate onset time of singularity in the boundary-layer theory. At this time, our penalty functions remain smaller than 1% of $u(\tau)$. Figure 5 shows that radial contraction has an opposite effect. Figure 6 shows the same effect demonstrated in Figs. 4 and 5 except that the cylinder translation is now not an impulsively started uniform motion but is given by a constant acceleration, followed by a constant velocity after some finite time τ_c. This latter cylinder motion is obviously more easily attainable in experiments.

VI. Discussion

An equation that describes the evolution of the separation angle on a deforming cylinder in an unsteady cross flow was derived from an iterative solution of the Navier-Stokes equation for finite Reynolds numbers. The first approximation gives an accurate description of flows in a thin layer near the cylinder.[6,7] The higher-order iterations successively extend the accurate description far into the outer flow. The validity of the iterative solution hinges on the condition that the penalty functions remain negligible. This can be verified a posteriori. These penalty functions were found to be negligible in a larger period of time for the case of constant acceleration than for the case of impulsively started motion. Thus, the time periods during which the iterative solution is valid can be ascertained a posteriori at each iteration. The advantage of the present method over other known methods is that only one computer program needs to be written for $u(\tau)$ and $a(\tau)$. The disadvantage of the present method is that the time period during which the iterative solution remains valid becomes increasingly small as R_e increases. The solutions of the evolution equation of the separation point on a circular cylinder reveal that a radial contraction retards separation due to the enhanced acceleration along the wall. The converse is true for the case of radial expansion. In the limit of $R_e \to 0$, it was shown that no separation at the wall other than that at the rear stagnation point can take place. All these results suggest that on the body of revolution translating at an angle of attack in a fluid, the locus of points at which the azimuthal component of skin friction changes sign, does not originate at the nose tip of the body but on the leeward ray somewhere downstream from the tip of the body. The results reported in this work were based on the second iterative solution. Hence, for the deforming cylinder only the primary flow separation points were found. For the follow-up work, we intend to investigate the onset and evolution of the symmetric secondary flow separation[21] at the deforming wall, as well as in the midstream and the growth of the closed wake, by use of the higher iterative solutions. It will be of great interest to find out from the higher iterative solutions if the boundary-layer singularity is linked to the appearance of secondary vortices.[24,25] However, it should be pointed out that singularity is associated with the boundary-layer equation, not with the Navier-Stokes equation. It may turn out that the secondary separation introduces no singularity in the higher-order iterated solution of the Navier-Stokes equations at finite R_e. This will be the case if the penalty functions remain small when the secondary separation occurs. The results of the second iteration suggest that this will be the case. It should also be pointed out that our obtained results show that the primary separation point first appears at the rear stagnation point and then migrates upstream along the cylinder surface. No secondary separation can take place during the initial period. This is consistent with the known numerical results[22] and the experiments.[17] The iterative solution of the form given by Eq. (28) for asymmetric flows will also be developed to study the breakup of a closed wake and the ensuing vortex shedding.

Appendix: A General Method of Superposition for Problems Involving a Moving Boundary

We wish to solve for the unknown function F from the following partial differential equation

$$(\partial_\tau - L)F(\boldsymbol{r},\tau) = G(\boldsymbol{r},\tau) \tag{A1}$$

with the boundary condition

$$F[\boldsymbol{a}(\tau),\tau] = 2u(\tau), \qquad \tau>0$$
$$= 0, \qquad \tau\leq 0 \tag{A2}$$

and the initial condition

$$F(\boldsymbol{r},\tau) = 0, \qquad \tau\leq 0 \tag{A3}$$

where τ is time, $\boldsymbol{r}$ a position vector, $G(\boldsymbol{r},\tau)$ a given function, L any linear differential operator in space variables, and $\boldsymbol{r}=\boldsymbol{a}(\tau)$ specifies the instantaneous position of the moving boundary at any time τ.

It is presently shown that the solution of this system is given by

$$F(\boldsymbol{r},\tau) = \int_0^\tau \partial_\tau F(\boldsymbol{r},\ell,\tau-\ell)\,\mathrm{d}\ell \tag{A4}$$

where $F(\boldsymbol{r},\ell,\tau)$ is the solution of Eq. (A1) subject to a fixed boundary condition

$$F[\boldsymbol{a}(\ell),\ell,\tau] = 2u(\ell)H(\tau), \qquad H(\tau) = \begin{cases} 1, & \tau>0 \\ 0, & \tau\leq 0 \end{cases} \tag{A5}$$

and the initial condition

$$F[\boldsymbol{r},\ell,\tau] = 0, \qquad \tau\leq 0 \tag{A6}$$

where ℓ is a constant parameter. Since the differential system (A1), (A5), and (A6) is invariant with respect to time translation, $F(\boldsymbol{r},\ell,\tau-\ell)$ and $F(\boldsymbol{r},\ell,\tau-\ell-\mathrm{d}\ell)$ are also solutions of Eq. (A1). Each satisfies Eq. (A5) and (A6) with τ in both of them replaced, respectively, by $\tau-\ell$ and $\tau-\ell-\mathrm{d}\ell$. It follows that $\mathrm{d}F$ given by

$$\mathrm{d}F = F(\boldsymbol{r},\ell,\tau-\ell) - F(\boldsymbol{r},\ell,\tau-\ell-\mathrm{d}\ell)$$
$$= \partial_T F(\boldsymbol{r},\ell,\tau-\ell)\,\mathrm{d}\ell \tag{A7}$$

satisfies Eq. (A1) since it is merely a linear superposition of two solutions of Eq. (A1). Moreover, Eq. (A7) satisfies the boundary condition

$$\mathrm{d}F[\boldsymbol{a}(\ell),\ell,\tau-\ell] = 2u(\ell)\,[H(\tau-\ell) - H(\tau-\ell-\mathrm{d}\ell)] \tag{A8}$$

and the initial condition

$$\mathrm{d}F = 0 \qquad \tau\leq\ell \tag{A9}$$

The integral of $\mathrm{d}F$ from $\ell=0$ to $\ell=\tau$, i.e.,

$$F(\boldsymbol{r},\tau) = \int_0^\tau \partial_\tau F(\boldsymbol{r},\ell,\tau-\ell)\,\mathrm{d}\ell \tag{A10}$$

being merely a linear superposition of solutions of Eq. (A1), each satisfying Eq. (A8) instantaneously from $\ell=0$ to $\ell=\tau$ for any $\tau>0$, is itself a solution of Eq. (A1). Moreover, it satisfies the boundary condition (A2) since the integration of Eq. (A8) gives

$$\int_0^\tau \mathrm{d}F[\boldsymbol{a}(\ell),\ell,\tau-\ell]\,\mathrm{d}\ell = F[\boldsymbol{a}(\ell),\tau]$$
$$= \int_0^\tau 2u(\ell)\,[H(\tau-\ell) - H(\tau-\ell-\mathrm{d}\ell)]\,\mathrm{d}\ell = 2u(\tau)$$

It also follows from Eqs. (A9) and (A10) and the fact that $\ell\geq 0$ that the initial condition (A3) is satisfied by Eq. (A10).

Note that Eq. (A2) is the Dirichlet boundary condition. It is obvious that the same method applies to the Neumann or mixed boundary-value problems when the boundary conditions (A2) and (A5) are replaced by boundary conditions specifying the spatial derivative of F or the linear combination of F and its spatial derivative.

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